Theory of Laminated Turbulence: Open Questions

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1 Introduction

The roots of the theory of nonlinear dispersive waves date back to hydrodynamics of the 19th century. It was observed, both experimentally and theoretically, that, under certain circumstances, the dissipative effects in nonlinear waves become less important then the dispersive ones. In this way, balance between nonlinearity and dispersion gives rise to formation of stable patterns (solitons, cnoidal waves, etc.). Driven by applications in plasma physics, these phenomena were widely studied, both analytically and numerically, starting from the middle of the 20th century. The main mathematical break-through in the theory of nonlinear evolutionary PDEs was the discovery of the phenomenon of their integrability that became the starting point of the modern theory of integrable systems with Korteweg-de Vries equation being the first instance in which integrability appeared. But most evolutionary PDEs are not integrable, of course. As a powerful tool for numerical simulations, the method of kinetic equation has been developed in 1960-th and applied to many different types of dispersive evolutionary PDEs. The wave kinetic equation is approximately equivalent to the initial nonlinear PDE: it is an averaged equation imposed on a certain set of correlation functions and it is in fact one limiting case of the quantum Bose-Einstein equation while the Boltzman kinetic equation is its other limit. Some statistical assumptions have been used in order to obtain kinetic equations; the limit of their applicability then is a very complicated problem which should be solved separately for each specific equation.

The role of the nonlinear dispersive PDEs in the theoretical physics is so important that the notion of dispersion is used for "physical" classification of the equations in partial variables. On the other hand, the only mentioning of the notion "dispersion relation" in mathematical literature we have found in the book of V.I.Arnold [1] who writes about important physical principles and concepts such as energy, variational principle, the Lagrangian theory, dispersion relations, the Hamiltonian formalism, etc. which gave a rise for the development of large areas in mathematics (theory of Fourier series and integrals, functional analysis, algebraic geometry and many others). But he also could not find place for it in the consequent mathematical presentation of the theory of PDEs and the words "dispersion relation" appear only in the introduction.

In our paper we present wave turbulence theory as a base of the "physical" classification of PDEs, trying to avoid as much as possible specific physical jargon and give a "pure" mathematician a possibility to follow its general ideas and results. We show that the main mathematical object of the wave turbulence theory is an algebraic system of equations called resonant manifolds. We also present here the model of the laminated wave turbulence that includes classical

statistical results on the turbulence as well as the results on the discrete wave systems. It is shown that discrete characteristics of the wave systems can be described in terms of integer points on the rational manifolds which is the main novelty of the theory of laminated turbulence. Some applications of this theory for explanation of important physical effects are given. A few open mathematical and numerical problems are formulated at the end. Our purpose is attract pure mathematicians to work on this subject.

2 General Notions

For the complicity of presentation we began this section with a very brief sketch of the traditional mathematical approach to the classification of PDEs.

2.1 Mathematical Classification

Well-known mathematical classification of PDEs is based on **the form of equations** and can be briefly presented as follows. For a bivariate PDE of the second order

$$a\psi_{xx} + b\psi_{xy} + c\psi_{yy} = F(x, y, \psi, \psi_x, \psi_y)$$

its characteristic equation is written as

$$\frac{\boldsymbol{d}x}{\boldsymbol{d}y} = \frac{b}{2a} \pm \frac{1}{2a}\sqrt{b^2 - 4ac}$$

and three types of PDEs are defined:

- $b^2 < 4ac$, elliptic PDE: $\psi_{xx} + \psi_{yy} = 0$
- $b^2 > 4ac$, hyperbolic PDE: $\psi_{xx} - \psi_{yy} - x\psi_x = 0$
- $b^2 = 4ac$, parabolic PDE: $\psi_{xx} - 2xy\psi_y - \psi = 0$

Each type of PDE demands then special type of initial/boundary conditions for the problem to be well-posed. "Bad" example of Tricomi equation $y\psi_{xx} + \psi_{yy} = 0$ shows immediately incompleteness of this classification even for second order PDEs because a PDE can change its type depending, for instance, on the initial conditions. This classification can be generalized to PDEs of more variables but not to PDEs of higher order.

2.2 Physical Classification

Physical classification of PDEs is based on **the form of solution** and is almost not known to pure mathematicians. In this case, a PDE is regarded in the very general form, without any restrictions on the number of variables or the order of equation. On the other hand, the necessary preliminary step in this classification is the division of all the variables into two groups - time- and space-like variables. This division originated from the special relativity theory where time and three-dimensional space are treated together as a single four-dimensional Minkowski space. In Minkowski space a metrics allowing to compute an interval s along a curve between two events is defined analogously to distance in Euclidean space:

$$\boldsymbol{d}s^2 = \boldsymbol{d}x^2 + \boldsymbol{d}y^2 + \boldsymbol{d}z^2 - c^2\boldsymbol{d}t^2$$

where c is speed of light, x, y, z and t denote respectively space and time variables. Notice that though in mathematical classification all variables are treated equally, obviously its results can be used in any applications only after similar division of variables have been done.

Suppose now that linear PDE with constant coefficients has a wave-like solution

$$\psi(x,t) = A \exp i[kx - øt] \quad \text{or} \quad \psi(x,t) = A \sin(kx - øt)$$

with amplitude A, wave-number k and wave frequency $ø$. Then the substitution of $\partial_t = -iø, \quad \partial_x = ik$ transforms LPDE into a **polynomial** on $ø$ and k, for instance:

$$\psi_t + \alpha\psi_x + \beta\psi_{xxx} = 0 \qquad \Rightarrow \; ø(k) = \alpha k - \beta k^3,$$

$$\psi_{tt} + \alpha^2\phi_{xxxx} = 0 \qquad \Rightarrow \; ø^2(k) = \alpha^2 k^4,$$

$$\psi_{tttt} - \alpha^2\psi_{xx} + \beta^2\psi = 0 \qquad \Rightarrow \; ø^4(k) = \alpha^2 k^2 + \beta^2$$

where α and β are constants.

Definition Real-valued function $ø = ø(k): \quad \mathrm{d}^2ø/\mathrm{d}k^2 \neq 0$ is called *dispersion relation* or *dispersion function.* A linear PDE with wave-like solutions are called evolutionary dispersive LPDE. A nonlinear PDE with dispersive linear part are called evolutionary dispersive NPDE.

This way all PDEs are divided into two classes - dispersive and non-dispersive [2]. This classification is not complementary to a standard mathematical one. For instance, though hyperbolic PDEs normally do not have dispersive wave solutions, the hyperbolic equation $\psi_{tt} - \alpha^2\psi_{xx} + \beta^2\psi = 0$ has them. Given dispersion relation allows to re-construct corresponding linear PDE. All definitions above could be easily reformulated for a case of more space variables, namely $x_1, x_2, ..., x_n$. Linear part of the initial PDE takes then form

$$P(\frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n})$$

and correspondingly dispersion relation can be computed from

$$P(-iø, ik_1, ..., ik_n) = 0$$

with the polynomial P. In this case we will have not a wave number k but a *wave vector* $\vec{k} = (k_1, ..., k_n)$ and the condition of non-zero second derivative of the dispersion function takes a matrix form:

$$|\frac{\partial^2\omega}{\partial k_i \partial k_j}| \neq 0.$$

2.3 Perturbation technique

Perturbation or asymptotic methods (see, for instance, [3]) are much in use in physics and are dealing with equations having some small parameter $\varepsilon > 0$. To understand the results presentated in the next Section one needs to have some clear idea about the perturbation technique and this is the reason why we give here a simple algebraic example of its application. The main idea of a perturbation method is very straightforward - an unknown solution, depending on ε,

is written out in a form of infinite series on different powers of ε and coefficients in front of any power of ε are computed consequently. Let us take an algebraic equation

$$x^2 - (3 - 2\varepsilon)x + 2 + \varepsilon = 0 \tag{1}$$

and try to find its asymptotic solutions. If $\varepsilon = 0$ we get

$$x^2 - 3x + 2 = 0 \tag{2}$$

with roots $x = 1$ and $x = 2$. Eq.(1) is called *perturbed* and Eq.(2) - *unperturbed.* Natural suggestion is that the solutions of perturbed equation differ only a little bit from the solutions of unperturbed one. Let us look for solutions of Eq.(1) in the form

$$x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + ...$$

where x_0 is a solution of Eq.(2), i.e. $x_0 = 1$ or $x_0 = 2$. Substituting this infinite series into Eq.(1), collecting all the terms with the same degree of ε and consequent equaling to zero all coefficients in front of different powers of ε leads to an algebraic system of equations

$$\begin{cases} \varepsilon^0 : x_0^2 - 3x_0 + 2 = 0, \\ \varepsilon^1 : 2x_0x_1 - 3x_1 - 2x_0 + 1 = 0, \\ \varepsilon^2 : 2x_0x_2 + x_1^2 - 3x_2 - 2x_1 = 0, \\ ... \end{cases} \tag{3}$$

with solutions

$$x_0 = 1, \quad x_1 = -1, \quad x_2 = 3, ... \quad \text{and} \quad x = 1 - \varepsilon + 3\varepsilon^2 + ...;$$

$$x_0 = 2, \quad x_1 = 3, \quad x_2 = -3, ... \quad \text{and} \quad x = 2 + 3\varepsilon - 3\varepsilon^2 + ...$$

Notice that exact solutions of Eq.(1) are

$$x = \frac{1}{2}[3 + 2\varepsilon \pm \sqrt{1 + 8\varepsilon + 4\varepsilon^2}]$$

and the use of binomial representation for the expression under the square root

$$(1 + 8\varepsilon + 4\varepsilon^2)^{\frac{1}{2}} = 1 + (8\varepsilon + 4\varepsilon^2) + \frac{\frac{1}{2}(\frac{-1}{2})}{2!}(8\varepsilon + 4\varepsilon^2)^2 + ... = 1 + 4\varepsilon - 6\varepsilon^2 + ...$$

gives finally

$$x = \frac{1}{2}(3 + 2\varepsilon - 1 - 4\varepsilon + 6\varepsilon^2 + ...) = 1 - \varepsilon + 3\varepsilon^2 + ...$$

and

$$x = \frac{1}{2}(3 + 2\varepsilon + 1 + 4\varepsilon - 6\varepsilon^2 + ...) = 2 + 3\varepsilon - 3\varepsilon^2 + ...$$

as before. This example was chosen because the exact solution in this case is known and can be compared to the asymptotic one. The same approach is used for partial differential equations, also in the cases when exact solutions are not known. The only difference would be more elaborated computations resulting in some system of ordinary differential equations instead of Eqs.(3) (see next Section).

3 Wave Turbulence Theory

3.1 Wave Resonances

Now we are going to introduce the notion of the wave resonance which is the mile-stone for the whole theory of evolutionary dispersive NPDE and therefore for the wave turbulence theory. Let us consider first a linear oscillator driven by a small force

$$x_{tt} + p^2 x = \varepsilon e^{i\Omega t}.$$

Here p is eigenfrequency of the system, Ω is frequency of the driving force and $\varepsilon > 0$ is a small parameter. Deviation of this system from equilibrium is small (of order ε), if there is no resonance between the frequency of the driving force $\varepsilon e^{i\Omega t}$ and an eigenfrequency of the system. If these frequencies coincide then the amplitude of oscillator grows linearly with the time and this situation is called *resonance* in physics. Mathematically it means *existence of unbounded solutions.*

Let us now regard a (weakly) nonlinear PDE of the form

$$L(\psi) = \varepsilon N(\psi) \tag{4}$$

where L is an arbitrary linear dispersive operator and N is an arbitrary nonlinear operator. Any two solutions of $L(\psi) = 0$ can be written out as

$$A_1 \exp i[\vec{k}_1\vec{x} - \omega(\vec{k}_1)t] \quad \text{and} \quad A_2 \exp i[\vec{k}_2\vec{x} - \omega(\vec{k}_2)t]$$

with constant amplitudes A_1, A_2. Intuitively natural expectation is that solutions of weakly nonlinear PDE will have the same form as linear waves but perhaps with amplitudes depending on time. Taking into account that nonlinearity is small, each amplitude is regarded as *a slow-varying function of time*, that is $A_j = A_j(t/\varepsilon)$. Standard notation is $A_j = A_j(T)$ where $T = t/\varepsilon$ is called *slow time.* Since wave energy is by definition proportional to amplitude's square A_j^2 it means that in case of nonlinear PDE *waves exchange their energy.* This effect is also described as "waves are interacting with each other" or "there exists energy transfer through the wave spectrum" or similar.

Unlike linear waves for which their linear combination was also solution of $L(\psi) = 0$, it is not the case for nonlinear waves. Indeed, substitution of two linear waves into the operator $\epsilon N(\psi)$ generates terms of the form $\exp i[(\vec{k}_1 + \vec{k}_2)\vec{x} - [\o(\vec{k}_1) + \omega(\vec{k}_2)]t$ which play the role of a small driving force for the linear wave system similar to the case of linear oscillator above. This driving force gives a small effect on a wave system till resonance occurs, i.e. till the wave number and the wave frequency of the driving force does not coincide with some wave number and some frequency of eigenfunction:

$$\begin{cases} \o_1 + \o_2 = \o_3, \\ \vec{k}_1 + \vec{k}_2 = \vec{k}_3 \end{cases} \tag{5}$$

where notation $\o_i = \omega(\vec{k}_i)$ is used. This system describes so-called *resonance conditions* or *resonance manifold.*

The perturbation technique described above produces the equations for the amplitudes of resonantly interacting waves $A_j = A_j(T)$. Let us demonstrate it taking as example barotropic vorticity equation (BVE) on a sphere

$$\frac{\partial \triangle \psi}{\partial t} + 2\frac{\partial \psi}{\partial \lambda} + \varepsilon J(\psi, \triangle \psi) = 0 \tag{6}$$

where

$$\triangle \psi = \frac{\partial^2 \psi}{\partial \phi^2} + \frac{1}{\cos^2 \phi}\frac{\partial^2 \psi}{\partial \lambda^2} - \tan\phi \frac{\partial \psi}{\partial \phi} \quad \text{and} \quad J(a,b) = \frac{1}{cos\phi}(\frac{\partial a}{\partial \lambda}\frac{\partial b}{\partial \phi} - \frac{\partial a}{\partial \phi}\frac{\partial b}{\partial \lambda}).$$

The linear part of spherical BVE has wave solutions in the form

$$AP_n^m(\sin\phi)\exp i[m\lambda + \frac{2m}{n(n+1)}t],$$

where A is constant wave amplitude, $\omega = -2m/[n(n+1)]$ and $P_n^m(x)$ is the associated Legendre function of degree n and order m.

One of the reasons to choose this equation as an example is following. Till now a linear wave was supposed to have much more simple form, namely, $A\exp i[\vec{k}\vec{x} - \omega(\vec{k})t]$ without any additional factor of a functional form. For a physicist it is intuitively clear that if the factor is some *oscillatory function of only space variables* then we will still have a wave of a sort "but it would be difficult to include it in an overall definition. We seem to be left at present with the looser idea that whenever oscillations in space are coupled with oscillation in time through a dispersion relation, we expect the typical effects of dispersive waves" [2]. By the way, most physically important dispersive equations have the waves of this form.

Now let us keep in mind that a wave is something more complicated then just a sin but still smooth and periodic, and let us look where perturbation method will lead us. An approximate solution has a form

$$\psi = \psi_0(\lambda,\phi,t,T) + \varepsilon\psi_1(\lambda,\phi,t,T) + \varepsilon^2\psi_2(\lambda,\phi,t,T) + ...$$

where $T = t/\varepsilon$ is the slow time and the zero approximation ψ_0 is given as a sum of three linear waves:

$$\psi_0(\lambda,\phi,t,T) = \sum_{k=1}^{3} A_k(T)P^{(k)}\exp(i\theta_k) \tag{7}$$

with notations $P^{(k)} = P_{n_k}^{m_k}$ and $\theta_k = m_k\lambda - \omega_k t$. Then

$$\begin{cases} \varepsilon^0: \quad \partial\triangle\psi_0/\partial t + 2\partial\psi_0/\partial\lambda = 0, \\ \varepsilon^1: \quad \partial\triangle\psi_1/\partial t + 2\partial\psi_1/\partial\lambda = -J(\psi_0,\triangle\psi_0) - \partial\triangle\psi_0/\partial T, \\ \varepsilon^2: ..... \end{cases} \tag{8}$$

and

$$\frac{\partial \triangle\psi_0}{\partial\lambda} = i\sum_{k=1}^{3} P^{(k)} m_k N_k[A_k \exp(i\theta_k) - A_k^* \exp(-i\theta_k)];$$

$$\frac{\partial \triangle\psi_0}{\partial\phi} = -\sum_{k=1}^{3} N_k \frac{d}{d\phi} P^{(k)} \cos\phi [A_k \exp(i\theta_k) + A_k^* \exp(-i\theta_k)];$$

$$J(\psi_0, \triangle\psi_0) =$$

$$-i\sum_{j,k=1}^{3} N_k m_j P^{(j)} \frac{d}{d\phi} P^{(k)} [A_j \exp(i\theta_j) - A_j^* \exp(-i\theta_j)][A_k \exp(i\theta_k) + A_k^* \exp(-i\theta_k)] +$$

$$+i\sum_{j,k=1}^{3} N_k m_k P^{(k)} \frac{d}{d\phi} P^{(j)} [A_j \exp(i\theta_j) + A_j^* \exp(-i\theta_j)][A_k \exp(i\theta_k) - A_k^* \exp(-i\theta_k)];$$

$$\frac{\partial \triangle\psi_0}{\partial T} = \sum_{k=1}^{3} P^{(k)} N_k [\frac{dA_k}{dT} \exp(i\theta_k) + \frac{dA_k^*}{dT} \exp(-i\theta_k)].$$

leads to the condition of unbounded growth of the left hand in the form

$$J(\psi_0, \triangle\psi_0) = \frac{\partial \triangle\psi_0}{\partial T}$$

with resonance conditions $\theta_j + \theta_k = \theta_i \quad \forall\ j,k,i = 1,2,3$. Let us fix some specific resonance condition, say, $\theta_1 + \theta_2 = \theta_3$, then

$$\frac{\partial \triangle\psi_0}{\partial T} \cong -N_3 P^{(3)} [\frac{dA_3}{dT} \exp(i\theta_3) + \frac{dA_3^*}{dT} \exp(-i\theta_3)],$$

$$J(\psi_0, \triangle\psi_0) \cong -i(N_1 - N_2)(m_2 P^{(2)} \frac{d}{d\phi} P^{(1)}) - m_1 P^{(1)} \frac{d}{d\phi} P^{(2)} \cdot$$
$$A_1 A_2 \exp[i(\theta_1 + \theta_2)] - A_1^* A_2^* \exp[-i(\theta_1 + \theta_2)],$$

where notation $\cong$ means that only those terms are written out which can generate chosen resonance. Let us substitute these expressions into the coefficient by ε^1, i.e. into the equation

$$\frac{\partial \triangle\psi_1}{\partial t} + 2\frac{\partial \psi_1}{\partial\lambda} = -J(\psi_0, \triangle\psi_0) - \frac{\partial \triangle\psi_0}{\partial T},$$

multiply both parts of it by

$$P^{(3)} \sin\phi [A_3 \exp(i\theta_3) + A_3^* \exp(-i\theta_3)]$$

and integrate all over the sphere with $t \to \infty$. As a result following equation can be obtained:

$$N_3 \frac{dA_3}{dT} = 2iZ(N_2 - N_1) A_1 A_2,$$

where

$$Z = \int_{-\pi/2}^{\pi/2} [m_2 P^{(2)} \frac{d}{d\phi} P^{(1)} - m_1 P^{(1)} \frac{d}{d\phi} P^{(2)}] \frac{d}{d\phi} P^{(3)} d\phi.$$

The same procedure obviously provides the analogous equations for A_2 and A_3 while fixing corresponding resonance conditions:

$$\begin{aligned} N_1 \frac{dA_1}{dT} &= -2iZ(N_2 - N_3)A_3A_2^*, \\ N_2 \frac{dA_2}{dT} &= -2iZ(N_3 - N_1)A_1^*A_3, \end{aligned} \tag{9}$$

In general, the simplest system of equations on the amplitudes of three resonantly interacting waves is often regarded in the form

$$\begin{cases} \dot{A}_1 = \alpha_1 A_3 A_2^*, \\ \dot{A}_2 = \alpha_2 A_1^* A_3, \\ \dot{A}_3 = \alpha_3 A_1 A_2 \,, \end{cases} \tag{10}$$

and is refereed to as a 3-wave system (keeping in mind that analogous system has to be written out for A_i^*). Coefficients α_i depend on the initial NPDE. Similar system of equations can be obtained for 4-wave interactions, with the products of three different amplitudes on the right hand, and so on.

3.2 Zakharov-Kolmogorov energy spectra

The idea that a dispersive wave system contains many resonances and wave interactions are stochastic led to the statistical theory of wave turbulence. This theory is well developed [4] and widely used in oceanology and plasma physics describing a lot of turbulent transport phenomena. Avoiding the language of Hamiltonian systems, correlators of a wave field, etc., one can formulate its main results in the following way. Any nonlinearity in Eq.(4) can be written out as

$$\Sigma_i \frac{V_{(12..i)} \delta(\vec{k}_1 + \vec{k}_2 + ... + \vec{k}_i)}{\delta(\o_1 + \o_2 + ... + \o_i)} A_1 A_2 \cdots A_i \tag{11}$$

where δ is Dirac delta-function and $V_{(12..i)}$ is a vertex coefficient. This presentation, together with some additional statistical suggestions, is used then to construct a wave kinetic equation, with corresponding vertex coefficients and delta-functions in the under-integral expression, of the form

$$\dot{A}_1 = \int |V_{(123)}|^2 \delta(\o_1 - \o_2 - \o_3) \delta(\vec{k}_1 - \vec{k}_2 - \vec{k}_3)(A_2A_3 - A_1A_2 - A_1A_3) \mathbf{d}\vec{k}_2 \mathbf{d}\vec{k}_3$$

for 3-waves interactions, and similar for i-waves interactions. One of the most important discoveries in the statistical wave turbulence theory are stationary exact solutions of the kinetic equations first found in [5]. These solutions are now called Zakharov-Kolmogorov (ZK) energy spectra and they describe energy cascade in the wave field. In other words, energy of the wave with wave vector $\vec{k}$ is proportional to k^α with $\alpha < 0$ and magnitude of α depends on the specific of the wave system. Discovery of ZK spectra played tremendous role in the wave turbulence theory and till the works of last decade [7] it was not realized that some turbulent effects are not due to the statistical properties of a wave field and are not described by kinetic equations or ZK energy spectra.

3.3 Small Divisors Problem

In order to use presentation (11) one has to check whether so defined nonlinearity is finite. This problem is known as *the small divisors problem* and its solution depends on whether wave vectors have real or integer coordinates.

Wave systems with *continuous spectra* were studied by Kolmogorov, Arnold and Moser [6] and main results of KAM-theory can be briefly formulated as follows. If dispersion function $\o$ is defined on real-valued wave vectors and the ratio $\alpha_{ij} = \o_i/\o_j$ *is not a rational number* for any two wave vectors $\vec{k}_i$ and $\vec{k}_j$, then

- (1C) Wave system is decomposed into disjoint invariant sets (KAM tori) carrying quasi-periodic motions;
- (2C) If the size of the wave system tends to infinity, (1C) does not contradict ergodicity, random phase approximation can be assumed, kinetic equations and ZK energy spectra describe the wave system properly;
- (3C) Union of invariant tori has positive Liouville measure and $\mathbb{Q}$ has measure 0, there exclusion of the waves with rational ratio of their dispersions is supposed to be not very important.

Wave systems with *discrete spectra* demonstrate [7] substantially different energetic behavior:

- (1D) Wave system is decomposed into disjoint discrete classes carrying periodic motions or empty; for any two waves with wave vectors $\vec{k}_i$ and $\vec{k}_j$ belonging to the same class, the ratio $\alpha_{ij} = \o_i/\o_j$ *is a rational number*;
- (2D) Energetic behavior of the wave system does not depend on its size, is not stochastic and is described by a few isolated periodic processes governed by Sys.(10);
- (3D) In many wave systems (for instance, in the systems with periodic or zero boundary conditions) *KAM-tori do not exist* and *discrete classes play the major role in the energy transfer*.

3.4 Laminated Wave Turbulence

The results formulated in the previous section gave rise to the model of laminated wave turbulence [9] which includes two co-existing layers of turbulence in a wave system - continuous and discrete layers, each demonstrating specific energetic behavior. In other words, KAM-theory describes the wave systems leaving some "holes" in the wave spectra which are "full-filled" in the theory of laminated turbulence.

Continuous layer, with its kinetic equation, energy cascades, ZK spectra, etc. is well-studied while the existence of the discrete layer was realized quite recently. In order to understand which manifestations of the discrete layer are to be expected in numerical or laboratory experiments, let us regard an example with dispersion function $\o = 1/\sqrt{m^2+n^2}$. First of all, it is important to realize: the fact that the ratio $\alpha_{ij} = \o_i/\o_j$ is a rational number *does not imply* that dispersion function $\o$ is a rational function. Indeed, for $\o = 1/\sqrt{m^2+n^2}$ we have

$$\o : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}$$

and for wave vectors $\vec{k}_1 = (2,1)$ and $\vec{k}_2 = (9,18)$, the ratio $\o_1/\o_2 = 1/3$ is rational number though $\o$ is irrational function of integer variables. Decomposition of all discrete waves into disjoint discrete classes Cl_q in this case has the form

$$\{\vec{k}_i = (m_i, n_i)\} \in Cl_q \ \text{ if } \ |\vec{k}_i| = \gamma_i \sqrt{q}, \ \forall \ i = 1, 2, ...$$

where γ_i is some integer and q is *the same square-free integer* for all wave vectors of the class Cl_q. The equation $\o_1 + \o_2 = \o_3$ has solutions only if

$$\exists \tilde{q} : \ \vec{k}_1, \ \vec{k}_2, \ \vec{k}_3 \in Cl_{\tilde{q}}.$$

This is *necessary condition*, not sufficient. The use of this necessary condition allows to cut back substantially computation time needed to find solution of irrational equations in integers. Namely, one has to construct classes first and afterwards look for the solutions among the waves belonging to the same class. In this way, instead of solving *the irrational equation on 6 variables*

$$1/\sqrt{m_1^2 + n_1^2} + 1/\sqrt{m_2^2 + n_2^2} = 1/\sqrt{m_3^2 + n_3^2} \tag{12}$$

it is enough to solve *the rational equation on 3 variables*

$$1/\gamma_1 + 1/\gamma_2 = 1/\gamma_3. \tag{13}$$

Some classes can be empty, for instance Cl_2 and Cl_3 in our example. Obviously, each non-empty class has infinite number of elements due to the existence of proportional vectors so that $\vec{k}_1 = (2,1)$ with the norm $|\vec{k}_1| = \sqrt{5}$ and its proportional $\vec{k}_2 = (9,18)$ with the norm $|\vec{k}_2| = 9\sqrt{5}$ belong to the same class Cl_5. On the other hand, not all the elements of a class are parts of some solution which means that *not all waves take part in resonant interactions.*

Let us come back to physical interpretation of these results. Resonantly interacting waves will change their amplitudes according to Sys.(10). In this case the role of ZK spectra k^α, $\alpha < 0$, is played by the interaction coefficient $Z \sim k^\alpha$, $\alpha > 0$ (see Fig.1). Non-interacting waves will have constant amplitudes (they are not shown in Fig.1). In the next Section we demonstrate some examples of different wave systems whose behavior is explained by the theory of laminated turbulence.

4 Examples

- **Ex.1** Turbulence of capillary waves (dispersion function $\omega^2 = k^3$, three-wave interactions) was studied in [10] in the frame of simplified dynamical equations for the potential flow of an ideal incompressible fluid. Coexistence of ZK energy spectra and a set of discrete waves with constant amplitudes was clearly demonstrated. The reason why in this case the amplitudes are constant is following: equation

  $$k_1^{3/2} + k_2^{3/2} = k_3^{3/2}$$

  has no integer solutions [7]. It means that there exist no three-wave resonant interactions among discrete capillary waves, they take no part in the energy transfer through the wave spectrum and just keep their energy at the low enough level of nonlinearity.

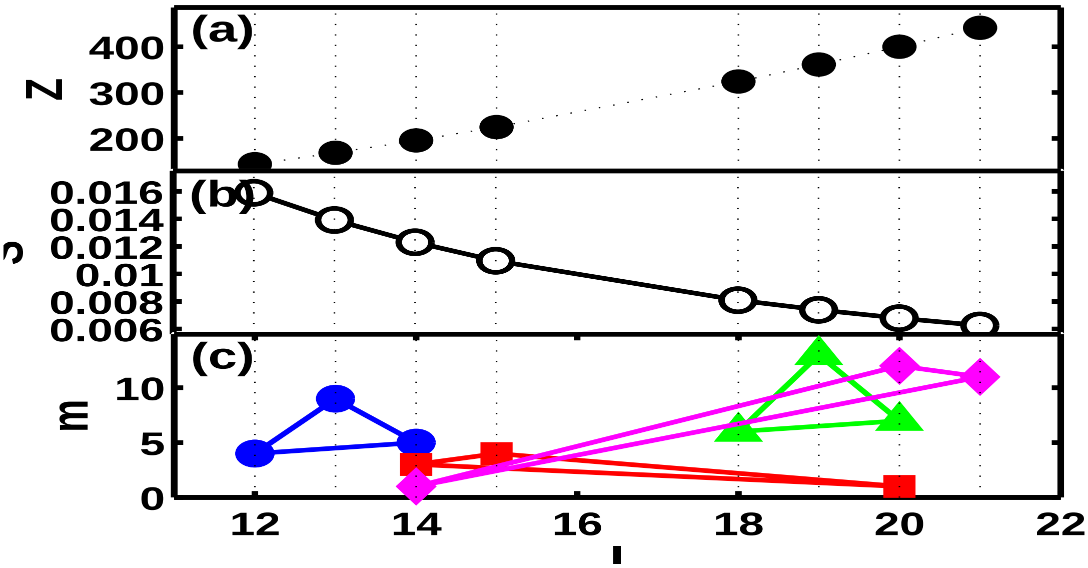

Figure 1: Two layers of turbulence are shown symbolically. Low panel: 2D-domain in spectral space, nodes of the integer lattice are connected with the lines which correspond to 3-wave resonant interactions of the discrete layer. Middle panel: ZK energy spectrum $k^{-3/2}$ with "the holes" in the nodes of the integer lattice. Upper panel: Interaction coefficient $Z \sim k^{3/2}$ in the nodes of the integer lattice.

- **Ex.2** Similar numerical simulations [11] with gravity waves on the surface of deep ideal incompressible fluid (dispersion function $\omega^4 = k^2$, four-wave interactions) show again coexistence of ZK energy spectra and a set of discrete waves. But in this case waves amplitudes are not constant any more, discrete waves do exchange their energy and in fact play major role in the energy transfer due to the fact that equation

  $$k_1^{1/2} + k_2^{1/2} = k_3^{1/2} + k_4^{1/2}$$

  has many non-trivial integer solutions [12] (it is important in this case that 2-dimensional waves are regraded, i.e. $k = |\vec{k}| = \sqrt{m^2 + n^2}$ with integer m, n).

- **Ex.3** Some recurrent patterns were found in different atmospheric data sets (rawindsonde time series of zonal wind, atmospheric angular momentum, atmospheric pressure, etc.) These large-scale quasi-periodic patterns appear repeatedly at fixed geographic locations, have periods 10-100 days and are called intra-seasonal oscillations in the Earth atmosphere. In [13] Eq.(6) (dispersion function $\omega = -2m/n(n+1)$, three-wave interactions) is studied which is classically regarded as a basic model of climate variability in the Earth atmosphere. It is shown that a possible explanation of the intra-seasonal oscillations can be done in terms of a few specific, resonantly interacting triads of planetary waves, isolated from the system of all the rest planetary waves.

  Remark In contrast to the first two examples, in this case *only discrete layer of turbulence exists.* Indeed, while $\o = -2m/n(n+1)$ is a rational function, any ratio $\o(m_i, n_i)/\o(m_j, n_j)$ is a rational number and KAM-theory is not applicable.

- **Ex.4** A very challenging idea indeed is to use the theory of laminated turbulence to explain so-called anomalous energy transport in tokamaks. Turbulent processes responsible for these effects are usually described as H- and L-modes and ELMs (high, low and

edge localized modes consequently). Interpretation of the known experimental results in terms of non-resonant (H), resonant (L) and resonant with small non-zero resonance width (ELM) modes gives immediately a lot of interesting results. In this case *non-resonant* discrete waves are of major interest because they will keep their energy as in **Ex.1**, for a substantial period of time. This approach allows to get two kind of results: 1) to describe the set of the boundary conditions providing no resonances at all - say, if the ratio of the sides in the rectangular domain is 2/7, no exact resonances exist; or 2) to compute explicitly all the characteristics of the non-resonant waves (wave numbers, frequencies, etc.) for given boundary conditions. Some preliminary results are presented in [14], in the frame of Hasegawa-Mima equation in a plane rectangular domain with zero boundary conditions (dispersion function $\omega = 1/\sqrt{n^2+m^2}$, three-wave interactions).

It is important to remember that a choice of initial and/or boundary conditions for a specific PDE might lead to a substantially different form of dispersion function and consequently to the qualitatively different behavior of the wave system. For instance, **Ex.3** and **Ex.4** are described by the same Eq.(6) regarded on a sphere (*rational dispersion function, only discrete layer of turbulence exists*) and in a rectangular (*irrational dispersion function, both layers exist*) respectively. For some equations, a special choice of boundary conditions leads to transcendental dispersion functions.

5 Open Questions

We have seen that the main algebraic object of the wave turbulence theory is the equation

$$\o(m_1,n_1)+\o(m_2,n_2)+...+\o(m_s,n_s)=0 \tag{14}$$

where dispersion function $\o$ is a solution of a dispersive evolutionary LPDE with $|\frac{\partial^2\omega}{\partial k_i \partial k_j}| \neq 0$. So defined class of dispersion functions includes rational, irrational or transcendental function, for instance

$$\o(k)=\alpha k-\beta k^3, \quad \o^4(k)=\alpha^2k^2+\beta^2, \quad \o=m/(k+1), \quad \o=\tanh \alpha k, \ \cdots$$

where α and β are constants and $k=\sqrt{m^2+n^2}$. Continuous layer of the wave turbulence, that is, with $m_i, n_i \in \mathbb{R}$, is well studied. On the contrary, there are still a lot of unanswered questions concerning the discrete layer of turbulence, $m_i, n_i \in \mathbb{Z}$, and we formulated here just a few of them.

Eq.(14) can be regarded as *a summation rule* for the rational points of the manifold given by $\o$. These manifolds have very special structure - namely, they can be transformed into a one-parametric family of simpler manifolds, namely (12) into (13). This situation is general enough, the definition of classes can be generalized for a given $c \in \mathbb{N}, c \neq 0, 1, -1$ considering algebraic numbers $k^{1/c}, k \in \mathbb{N}$ and their unique representation

$$k_c = \gamma q^{1/c}, \gamma \in \mathbb{Z}$$

where q is a product

$$q = p_1^{e_1} p_2^{e_2} ... p_n^{e_n},$$

while $p_1, ...p_n$ are all different primes and the powers $e_1, ...e_n \in \mathbb{N}$ are all smaller than c. Then algebraic numbers with the same q form the class Cl_q and the following statement holds:

The equation $a_1k_1 + a_2k_2 ... + a_nk_n = 0, a_i \in \mathbb{Z}$ *where each* $k_i = \gamma_i q_i^{1/c}$ *belongs to some class* $q_i \in q_1, q_2 ... q_l, \quad l < n$ *with ... is equivalent to a system*

$$\begin{cases} a_{q_1,1}\gamma_{q_1,1} + a_{q_1,2}\gamma_{q_1,2} + ... + a_{q_1,n_1}\gamma_{q_1,n_1} = 0 \\ a_{q_2,1}\gamma_{q_2,1} + a_{q_2,2}\gamma_{q_2,2} + ... + a_{q_2,n_2}\gamma_{q_2,n_2} = 0 \\ ... \\ a_{q_l,1}\gamma_{q_l,1} + a_{q_l,2}\gamma_{q_l,2} + ... + a_{q_l,nl}\gamma_{q_l,nl} = 0 \end{cases} \tag{15}$$

The questions are: what is the geometry underlying this parametrization? What is known about these sort of manifolds? What other properties of the resonance manifold are defined by a given summation rule? What additional information about these manifolds gives us the fact that they have many (often infinitely many) integer points?

Another group of questions concerns transcendental dispersion functions. All our examples were constructed for rational and irrational dispersion functions and the theoretical results were based on some classical theorems on the linear independence of some sets of algebraic numbers. In the case of a transcendental dispersion function like $\o = \tanh \alpha k$ similar reasoning can be carried out using the theorem on the linear independence of the exponents but it is not done yet. The question about special functions in this context is completely unexplored though very important. For instance, a dispersion function for capillary waves in a circle domain is described by Bessel function. Any results on their resonant interactions will shed some light on the nature of Faraday instability.

One of the most interesting questions about the resonance manifolds would be to study their invariants, i.e. some new function $f = f(m, n, \o)$ such that $\o_1 + \o_2 = \o_3$ implies $f_1 + f_2 = f_3$. An example of this sort of analysis is given in [15] for 3-wave interactions of drift waves with $\o = \alpha x/(1 + y^2)$ but for real-valued wave vectors, $x, y \in \mathbb{R}$. Existence of the invariants is important because it is directly connected with the integrability of corresponding nonlinear PDE. Coming back to the physical language this means that the wave system possesses some additional conservation law.

A very important task would be to develop fast algorithms to compute integer points on the resonance manifolds. The parametrization property allows to construct specific algorithms for a given dispersion function as it was done in [12] for 4-wave interactions of gravity waves, $\o = (m^2 + n^2)^{1/4}$. The work on the generic algorithm for a dispersion function $\o = \o(k)$ ($k = \sqrt{m^2 + n^2}$ or $k = \sqrt{m^2 + n^2 + l^2}$ with integer m, n, l) is on the way [16] but it does not cover even simple cases like $\o = m/n^2$ not mentioning transcendental dispersion functions.

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